

# WEIGHTED VECTOR-VALUED BOUNDS FOR A CLASS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS AND APPLICATIONS

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**ABSTRACT.** In this paper, we investigate the weighted vector-valued bounds for a class of multilinear singular integral operators, and its commutators, from  $L^{p_1}(l^{q_1}; \mathbb{R}^n, w_1) \times \cdots \times L^{p_m}(l^{q_m}; \mathbb{R}^n, w_m)$  to  $L^p(l^q; \mathbb{R}^n, \nu_{\vec{w}})$ , with  $p_1, \dots, p_m, q_1, \dots, q_m \in (1, \infty)$ ,  $1/p = 1/p_1 + \cdots + 1/p_m$ ,  $1/q = 1/q_1 + \cdots + 1/q_m$  and  $\vec{w} = (w_1, \dots, w_m)$  a multiple  $A_{\vec{p}}$  weights. Our argument also leads to the weighted weak type endpoint estimates for the commutators.

## 1. INTRODUCTION

In his remarkable work [32], Muckenhoupt characterized the class of weights  $w$  such that  $M$ , the Hardy-Littlewood maximal operator, satisfies the weighted  $L^p$  ( $p \in (1, \infty)$ ) estimate

$$(1.1) \quad \|Mf\|_{L^{p,\infty}(\mathbb{R}^n, w)} \lesssim \|f\|_{L^p(\mathbb{R}^n, w)}.$$

The inequality (1.1) holds if and only if  $w$  satisfies the  $A_p(\mathbb{R}^n)$  condition, that is,

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(x) dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes in  $\mathbb{R}^n$ ,  $[w]_{A_p}$  is called the  $A_p$  constant of  $w$ . Also, Muckenhoupt proved that  $M$  is bounded on  $L^p(\mathbb{R}^n, w)$  if and only if  $w$  satisfies the  $A_p(\mathbb{R}^n)$  condition. Since then, considerable attention has been paid to the theory of  $A_p(\mathbb{R}^n)$  and the weighted norm inequalities with  $A_p(\mathbb{R}^n)$  weights for main operators in Harmonic Analysis, see [18, Chapter 9] and related references therein.

However, the classical results on the weighted norm inequalities with  $A_p(\mathbb{R}^n)$  weights did not reflect the quantitative dependence of the  $L^p(\mathbb{R}^n, w)$  operator norm in terms of the relevant constant involving the weights. The question of the sharp dependence of the weighted estimates in terms of the  $A_p(\mathbb{R}^n)$  constant specifically raised by Buckley [3], who proved that if  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^n)$ , then

$$(1.2) \quad \|Mf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n, w)}.$$

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Moreover, the estimate (1.2) is sharp since the exponent  $1/(p-1)$  can not be replaced by a smaller one. Hytönen and Pérez [25] improved the estimate (1.4), and showed that

$$(1.3) \quad \|Mf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} ([w]_{A_p} [w^{-\frac{1}{p-1}}]_{A_\infty})^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n, w)}.$$

where and in the following, for a weight  $u$ ,  $[u]_{A_\infty}$  is defined by

$$[u]_{A_\infty} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u\chi_Q)(x) dx.$$

It is well known that for  $w \in A_p(\mathbb{R}^n)$ ,  $[w^{-\frac{1}{p-1}}]_{A_\infty} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$ . Thus, (1.3) is more subtle than (1.2).

The sharp dependence of the weighted estimates of singular integral operators in terms of the  $A_p(\mathbb{R}^n)$  constant was much more complicated. Petermichl [35, 36] solved this question for Hilbert transform and Riesz transform. Hytönen [23] proved that for a Calderón-Zygmund operator  $T$  and  $w \in A_2(\mathbb{R}^n)$ ,

$$(1.4) \quad \|Tf\|_{L^2(\mathbb{R}^n, w)} \lesssim_n [w]_{A_2} \|f\|_{L^2(\mathbb{R}^n, w)}.$$

This solved the so-called  $A_2$  conjecture. Combining the estimate (1.4) and the extrapolation theorem in [12], we know that for a Calderón-Zygmund operator  $T$ ,  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^n)$ ,

$$(1.5) \quad \|Tf\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\mathbb{R}^n, w)}.$$

In [26], Lerner gave a much simpler proof of (1.5) by controlling the Calderón-Zygmund operator using sparse operators.

Let  $K(x; y_1, \dots, y_m)$  be a locally integrable function defined away from the diagonal  $x = y_1 = \dots = y_m$  in  $\mathbb{R}^{mn}$ . An operator  $T$  defined on  $\mathcal{S}(\mathbb{R}^n) \times \dots \times \mathcal{S}(\mathbb{R}^n)$  (Schwartz space) and taking values in  $\mathcal{S}'(\mathbb{R}^n)$ , is said to be an  $m$ -multilinear singular integral operator with kernel  $K$ , if  $T$  is  $m$ -multilinear, and satisfies that

$$(1.6) \quad T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} K(x; y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j) dy_1 \dots dy_m,$$

for bounded functions  $f_1, \dots, f_m$  with compact supports, and  $x \in \mathbb{R}^n \setminus \bigcap_{j=1}^m \text{supp } f_j$ . Operators of this type were originated in the remarkable works of Coifman and Meyer [8], [9], and are useful in multilinear analysis. We say that  $T$  is an  $m$ -linear Calderón-Zygmund operator, if  $T$  is bounded from  $L^{r_1}(\mathbb{R}^n) \times \dots \times L^{r_m}(\mathbb{R}^n)$  to  $L^r(\mathbb{R}^n)$  for some  $r_1, \dots, r_m \in (1, \infty)$  and  $r \in (1/m, \infty)$  with  $1/r = 1/r_1 + \dots + 1/r_m$ , and  $K$  is a multilinear Calderón-Zygmund kernel, that is,  $K$  satisfies the size condition that for all  $(x, y_1, \dots, y_m) \in \mathbb{R}^{(m+1)n}$  with  $x \neq y_j$  for some  $1 \leq j \leq m$ ,

$$(1.7) \quad |K(x; y_1, \dots, y_m)| \lesssim \frac{1}{\left(\sum_{j=1}^m |x - y_j|\right)^{mn}}$$

and satisfies the regularity condition that for some  $\alpha \in (0, 1]$

$$|K(x; y_1, \dots, y_m) - K(x'; y_1, \dots, y_m)| \lesssim \frac{|x - x'|^\alpha}{\left(\sum_{j=1}^m |x - y_j|\right)^{mn+\alpha}}$$

whenever  $\max_{1 \leq k \leq m} |x - y_k| \geq 2|x - x'|$ , and for all  $1 \leq j \leq m$ ,

$$|K(x; y_1, \dots, y_j, \dots, y_m) - K(x; y_1, \dots, y'_j, \dots, y_m)| \lesssim \frac{|y_j - y'_j|^\alpha}{(\sum_{i=1}^m |x - y_i|)^{mn+\alpha}}$$

whenever  $\max_{1 \leq k \leq m} |x - y_k| \geq 2|y_j - y'_j|$ . Grafakos and Torres [19] considered the behavior of multilinear Calderón-Zygmund operators on  $L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$ , and established a  $T1$  type theorem for the operator  $T$ . To consider the weighted estimates for the multilinear Calderón-Zygmund operators, Lerner, Ombrossi, Pérez, Torres and Trojillo-Gonzalez [27] introduced the following definition.

**Definition 1.1.** Let  $m \in \mathbb{N}$ ,  $w_1, \dots, w_m$  be weights,  $p_1, \dots, p_m \in [1, \infty)$ ,  $p \in (0, \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_m$ . Set  $\vec{w} = (w_1, \dots, w_m)$ ,  $\vec{P} = (p_1, \dots, p_m)$  and  $\nu_{\vec{w}} = \prod_{k=1}^m w_k^{p/p_k}$ . We say that  $\vec{w} \in A_{\vec{P}}(\mathbb{R}^{mn})$  if

$$[\vec{w}]_{A_{\vec{P}}} = \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \nu_{\vec{w}}(x) dx \right) \prod_{k=1}^m \left( \frac{1}{|Q|} \int_Q w_k^{-\frac{1}{p_k-1}}(x) dx \right)^{p/p'_k} < \infty,$$

when  $p_k = 1$ ,  $\left( \frac{1}{|Q|} \int_Q w_k^{-\frac{1}{p_k-1}}(x) dx \right)^{1-1/p_k}$  is understood as  $(\inf_Q w_k)^{-1}$ .

Lerner et al. [27] proved that if  $p_1, \dots, p_m \in [1, \infty)$  and  $p \in [1/m, \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_m$ , and  $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$ , then an  $m$ -linear Calderón-Zygmund operator  $T$  is bounded from  $L^{p_1}(\mathbb{R}^n, w_1) \times \dots \times L^{p_m}(\mathbb{R}^n, w_m)$  to  $L^{p, \infty}(\mathbb{R}^n, \nu_{\vec{w}})$ , and when  $\min_{1 \leq j \leq m} p_j > 1$ ,  $T$  is bounded from  $L^{p_1}(\mathbb{R}^n, w_1) \times \dots \times L^{p_m}(\mathbb{R}^n, w_m)$  to  $L^p(\mathbb{R}^n, \nu_{\vec{w}})$ . Li, Moen and Sun [30] considered the sharp dependence of the weighted estimates of multilinear Calderón-Zygmund operators in terms of the  $A_{\vec{P}}(\mathbb{R}^{mn})$  constant, and proved that

**Theorem 1.2.** Let  $T$  be an  $m$ -linear Calderón-Zygmund operator,  $p_1, \dots, p_m \in (1, \infty)$ ,  $p \in [1, \infty)$  such that  $1/p = 1/p_1 + \dots + 1/p_m$ ,  $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$ . Then

$$(1.8) \quad \|T(f_1^k, \dots, f_m^k)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}.$$

Moreover, the exponent on  $[\vec{w}]_{A_{\vec{P}}}$  is sharp.

Conde-Alongso and Rey [7] proved that the conclusion in Theorem 1.2 is still true for the case  $p \in (1/m, 1)$ . For other works about the weighted estimates of multilinear Calderón-Zygmund operators, see [31, 1, 11] and references therein.

To consider the mapping properties for the commutator of Calderón, Duong, Grafakos and Yan [14] introduced a class of multilinear singular integral operators via the following generalized approximation to the identity.

**Definition 1.3.** A family of operators  $\{A_t\}_{t>0}$  is said to be an approximation to the identity, if for every  $t > 0$ ,  $A_t$  can be represented by the kernel at in the following sense: for every function  $u \in L^p(\mathbb{R}^n)$  with  $p \in [1, \infty]$  and a. e.  $x \in \mathbb{R}^n$ ,

$$A_t u(x) = \int_{\mathbb{R}^n} a_t(x, y) u(y) dy,$$

and the kernel  $a_t$  satisfies that for all  $x, y \in \mathbb{R}^n$  and  $t > 0$ ,

$$(1.9) \quad |a_t(x, y)| \leq h_t(x, y) = t^{-n/s} h\left(\frac{|x - y|}{t^{1/s}}\right),$$

where  $s > 0$  is a constant and  $h$  is a positive, bounded and decreasing function such that for some constant  $\eta > 0$ ,

$$(1.10) \quad \lim_{r \rightarrow \infty} r^{n+\eta} h(r) = 0.$$

**Assumption 1.4.** For each fixed  $j$  with  $1 \leq j \leq m$ , there exists an approximation to the identity  $\{A_t^j\}_{t>0}$  with kernels  $\{a_t^j(x, y)\}_{t>0}$ , and there exist kernels  $K_t^j(x; y_1, \dots, y_m)$ , such that for bounded functions  $f_1, \dots, f_m$  with compact supports, and  $x \in \mathbb{R}^n \setminus \cap_{k=1}^m \text{supp } f_k$ ,

$$T(f_1, \dots, f_{j-1}, A_t^j f_j, f_{j+1}, \dots, f_m)(x) = \int_{\mathbb{R}^{nm}} K_t^j(x; y_1, \dots, y_m) \prod_{k=1}^m f_k(y_k) d\vec{y},$$

and there exists a function  $\phi \in C(\mathbb{R})$  with  $\text{supp } \phi \subset [-1, 1]$ , and a constant  $\varepsilon \in (0, 1]$ , such that for all  $x, y_1, \dots, y_m \in \mathbb{R}^n$  and all  $t > 0$  with  $2t^{1/s} \leq |x - y_j|$ ,

$$\begin{aligned} & |K(x; y_1, \dots, y_m) - K_t^j(x; y_1, \dots, y_m)| \\ & \lesssim \frac{t^{\varepsilon/s}}{(\sum_{k=1}^m |x - y_k|)^{mn+\varepsilon}} + \frac{1}{(\sum_{k=1}^m |x - y_k|)^{mn}} \sum_{1 \leq i \leq m, i \neq j} \phi\left(\frac{|y_i - y_j|}{t^{1/s}}\right). \end{aligned}$$

As it was pointed out in [14], an operator with such a kernel is called a multilinear singular integral operator with non-smooth kernel, since the kernel  $K$  may enjoy no smoothness in the variables  $y_1, \dots, y_m$ . Also, it was pointed out in [14] that if  $T$  is an  $m$ -linear Calderón-Zygmund operator, then  $T$  also satisfies Assumption 1.4. Duong, Grafakos and Yan [14] proved that if  $T$  satisfies Assumption 1.4, and is bounded from  $L^{r_1}(\mathbb{R}^n) \times \dots \times L^{r_m}(\mathbb{R}^n)$  to  $L^{r, \infty}(\mathbb{R}^n)$  for some  $r_1, \dots, r_m \in (1, \infty)$  and  $r \in (1/m, \infty)$  with  $1/r = 1/r_1 + \dots + 1/r_m$ , then  $T$  is also bounded from  $L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$  to  $L^{1/m, \infty}(\mathbb{R}^n)$ . Recently, Hu and Li [21] considered the mapping properties from  $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$  to  $L^{1/m, \infty}(l^q; \mathbb{R}^n)$  for the multilinear operator which satisfies Assumption 1.4.

The first purpose of this paper is to give an extension of Theorem 1.2 to the operators satisfying Assumption 1.4. We further assume the kernel  $K$  satisfies the following regularity condition: for  $x, x', y_1, \dots, y_m \in \mathbb{R}^n$  with  $8|x - x'| < \min_{1 \leq j \leq m} |x - y_j|$ , and each number  $D$  such that  $2|x - x'| < D$  and  $4D < \min_{1 \leq j \leq m} |x - y_j|$ ,

$$(1.11) \quad |K(x; y_1, \dots, y_m) - K(x'; y_1, \dots, y_m)| \lesssim \frac{D^\gamma}{(\sum_{j=1}^m |x - y_j|)^{nm+\gamma}}.$$

This condition was introduced in [22], in order to established the weighted estimates for multilinear singular integral operators with non-smooth kernels. As it was pointed out in [22], the operators considered in [13, 17] also satisfies Assumption 1.4 and (1.11). On the other hand, it is obvious that if  $T$  is an  $m$ -linear Calderón-Zygmund operator, then  $T$  also satisfies (1.11). Thus, the operators we consider here contain multilinear Calderón-Zygmund operators and multilinear singular integral operators with non-smooth kernels. To state our results, we first recall some notations.

Let  $p, r \in (0, \infty]$  and  $w$  be a weight. As usual, for a sequence of numbers  $\{a_k\}_{k=1}^\infty$ , we denote  $\|\{a_k\}\|_{l^r} = (\sum_k |a_k|^r)^{1/r}$ . The space  $L^p(l^r; \mathbb{R}^n, w)$  is defined as

$$L^p(l^r; \mathbb{R}^n, w) = \{\{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)} < \infty\}$$

where

$$\|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} \|\{f_k(x)\}\|_{l^r}^p w(x) dx \right)^{1/p}.$$

The space  $L^{p, \infty}(l^r; \mathbb{R}^n, w)$  is defined as

$$L^{p, \infty}(l^r; \mathbb{R}^n, w) = \{\{f_k\}_{k=1}^\infty : \|\{f_k\}\|_{L^{p, \infty}(l^r; \mathbb{R}^n, w)} < \infty\}$$

with

$$\|\{f_k\}\|_{L^{p, \infty}(l^r; \mathbb{R}^n, w)}^p = \sup_{\lambda > 0} \lambda^p w\left(\left\{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{l^r} > \lambda\right\}\right).$$

When  $w \equiv 1$ , we denote  $\|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n, w)}$  ( $\|\{f_k\}\|_{L^{p, \infty}(l^r; \mathbb{R}^n, w)}$ ) by  $\|\{f_k\}\|_{L^p(l^r; \mathbb{R}^n)}$  ( $\|\{f_k\}\|_{L^{p, \infty}(l^r; \mathbb{R}^n)}$ ) for simplicity.

Our first result can be stated as follows.

**Theorem 1.5.** *Let  $m \geq 2$ ,  $T$  be an  $m$ -linear operator with kernel  $K$  in the sense of (1.6),  $r_1, \dots, r_m \in (1, \infty)$ ,  $r \in (0, \infty)$  such that  $1/r = 1/r_1 + \dots + 1/r_m$ . Suppose that*

- (i)  *$T$  is bounded from  $L^{r_1}(\mathbb{R}^n) \times \dots \times L^{r_m}(\mathbb{R}^n)$  to  $L^r(\mathbb{R}^n)$ ;*
- (ii) *The kernel  $K$  satisfies size condition (1.7) and regular condition (1.11);*
- (iii)  *$T$  satisfies the Assumption 1.4.*

*Let  $p_1, \dots, p_m, q_1, \dots, q_m \in (1, \infty)$ ,  $p, q \in (\frac{1}{m}, \infty)$  such that  $1/p = 1/p_1 + \dots + 1/p_m$ ,  $1/q = 1/q_1 + \dots + 1/q_m$ ,  $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$ . Then*

$$(1.12) \quad \|\{T(f_1^k, \dots, f_m^k)\}\|_{L^p(l^q; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \prod_{j=1}^m \|\{f_j^k\}\|_{L^{p_j}(l^{q_j}; \mathbb{R}^n, w_j)}.$$

**Remark 1.6.** As we pointed out, operators in Theorem 1.5 contain multilinear Calderón-Zygmund operators as examples. This, together with the examples in [30], shows that the estimate (1.12) is sharp.

Now let  $b$  be a locally integrable function. For  $1 \leq j \leq m$ , define the commutator  $[b, T]_j$  by

$$[b, T]_j(\vec{f})(x) = b(x)T(f_1, \dots, f_m)(x) - T(f_1, \dots, f_{j-1}, bf_j, f_{j+1}, \dots, f_m)(x).$$

Let  $b_1, \dots, b_m$  be locally integrable functions and  $\vec{b} = (b_1, \dots, b_m)$ . The multilinear commutator of  $T$  and  $\vec{b}$  is defined by

$$(1.13) \quad T_{\vec{b}}(f_1, \dots, f_m)(x) = \sum_{j=1}^m [b_j, T]_j(f_1, \dots, f_m)(x).$$

As it was showed in [6, 25, 11], by the conclusion (1.12), we can prove that, under the hypothesis of Theorem 1.5, for  $p_1, \dots, p_m, p \in (1, \infty)$  and  $\vec{w} \in A_{\vec{P}}(\mathbb{R}^{mn})$ ,

$$(1.14) \quad \begin{aligned} \|T_{\vec{b}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} &\lesssim \|\vec{b}\|_{\text{BMO}(\mathbb{R}^n)} [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \\ &\quad \times ([\nu_{\vec{w}}]_{A_\infty} + \sum_{j=1}^m [\sigma_j]_{A_\infty}) \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}. \end{aligned}$$

However, for the case of  $p \in (0, 1)$ , we do not know if we can deduce the weighted estimate for  $T_{\vec{b}}$  like (1.14) from (1.12), the argument used in [6, 25, 11] does not apply.

**Definition 1.7.** Let  $s \in [1, \infty)$ . A measurable function  $b$  on  $\mathbb{R}^n$  is said to belong to the space  $Osc_{\exp L^s}(\mathbb{R}^n)$ , if  $\|b\|_{Osc_{\exp L^s}(\mathbb{R}^n)} < \infty$ , with

$$\|b\|_{Osc_{\exp L^s}(\mathbb{R}^n)} = \inf \left\{ C > 0 : \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \exp \left( \frac{|b(x) - \langle b \rangle_Q|}{C} \right)^s dx \leq 2 \right\},$$

where and in the following,  $\langle b \rangle_Q = \frac{1}{|Q|} \int_Q b(y) dy$ .

For details of this space, see [33]. We remark that  $Osc_{\exp L^1}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ .

Our result concerning the weighted bound of  $T_{\vec{b}}$  can be stated as follows.

**Theorem 1.8.** *Let  $T$  be an  $m$ -linear operator as in Theorem 1.5 and  $T_{\vec{b}}$  the commutator defined by (1.13). Then for  $p_1, \dots, p_m; q_1, \dots, q_m \in (1, \infty)$ ,  $p, q \in (1/m, \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_m$  and  $1/q = 1/q_1 + \dots + 1/q_m$ , and  $\vec{w} \in A_{\vec{p}}(\mathbb{R}^{mn})$ ,*

$$(1.15) \quad \begin{aligned} \|\{T_{\vec{b}}(f_1^k, \dots, f_m^k)\}\|_{L^p(l^q; \mathbb{R}^n, \nu_{\vec{w}})} &\lesssim \left( \sum_{j=1}^m \|b_j\|_{Osc_{\exp L^{s_j}}(\mathbb{R}^n)} \right) [\vec{w}]_{A_{\vec{p}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \\ &\times \left( [\nu_{\vec{w}}]_{A_{\infty}}^{\frac{1}{s_*}} + \sum_{i=1}^m [\sigma_i]_{A_{\infty}}^{\frac{1}{s_i}} \right) \prod_{j=1}^m \|f_j^k\|_{L^{p_j}(l^{q_j}; \mathbb{R}^n, w_j)}, \end{aligned}$$

where and in the following,  $\sigma_j(x) = w_j^{-\frac{1}{p_j-1}}(x)$ ,  $s_* = \min_{1 \leq i \leq m} s_i$ .

Our argument in the proof of Theorems 1.5 and 1.8 also leads to the following weighted weak type endpoint estimate of  $T_{\vec{b}}$ .

**Theorem 1.9.** *Let  $T$  be an  $m$ -linear operator in Theorem 1.5,  $b_j \in Osc_{\exp L^{s_j}}(\mathbb{R}^n)$  ( $j = 1, \dots, m$ ) and  $T_{\vec{b}}$  be the commutator defined by (1.13). Then for  $q_1, \dots, q_m \in (1, \infty)$ ,  $q \in (1/m, \infty)$  with  $1/q = 1/q_1 + \dots + 1/q_m$ ,  $\vec{w} \in A_{1, \dots, 1}(\mathbb{R}^{mn})$  and  $\lambda > 0$ ,*

$$(1.16) \quad \begin{aligned} \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \|\{T_{\vec{b}}(f_1^k, \dots, f_m^k)(x)\}\|_{l^q} > \lambda\}) \\ \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \frac{\|\{f_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m}}} \log^{\frac{1}{s_*}} \left( 1 + \frac{\|\{f_j^k(y_j)\}\|_{l^{q_j}}}{\lambda^{\frac{1}{m}}} \right) w_j(y_j) dy_j \right)^{\frac{1}{m}}. \end{aligned}$$

**Remark 1.10.** For the case that  $T$  is multilinear Calderón-Zygmund operator and  $b_1, \dots, b_m \in BMO(\mathbb{R}^n)$ , (1.16) (the case  $\{f_j^k\} = \{f_j\}$ ) was proved in [27]. Although Bui and Duong [2] considered the weighted estimate for  $T_{\vec{b}}$  under the hypothesis of Theorem 1.5, the argument in [27] does not leads to the conclusion in Theorem 1.9.

In what follows,  $C$  always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol  $A \lesssim B$  to denote that there exists a positive constant  $C$  such that  $A \leq CB$ . Constant with subscript such as  $C_1$ , does not change in different occurrences. For any set  $E \subset \mathbb{R}^n$ ,  $\chi_E$  denotes its characteristic function. For a cube  $Q \subset \mathbb{R}^n$  and  $\lambda \in (0, \infty)$ , we use  $\ell(Q)$  ( $\text{diam} Q$ ) to denote the side length (diameter) of  $Q$ , and  $\lambda Q$  to denote the cube with the same center as  $Q$  and whose side length is  $\lambda$  times that of  $Q$ . For  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $B(x, r)$  denotes the ball centered at  $x$  and having radius  $r$ .

## 2. PRELIMINARIES

Recall that the standard dyadic grid in  $\mathbb{R}^n$  consists of all cubes of the form

$$2^{-k}([0, 1)^n + j), \quad k \in \mathbb{Z}, \quad j \in \mathbb{Z}^n.$$

Denote the standard dyadic grid by  $\mathcal{D}$ . For a fixed cube  $Q$ , denote by  $\mathcal{D}(Q)$  the set of dyadic cubes with respect to  $Q$ , that is, the cubes from  $\mathcal{D}(Q)$  are formed by repeating subdivision of  $Q$  and each of descendants into  $2^n$  congruent subcubes.

As usual, by a general dyadic grid  $\mathcal{D}$ , we mean a collection of cube with the following properties: (i) for any cube  $Q \in \mathcal{D}$ , its side length  $\ell(Q)$  is of the form  $2^k$  for some  $k \in \mathbb{Z}$ ; (ii) for any cubes  $Q_1, Q_2 \in \mathcal{D}$ ,  $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$ ; (iii) for each  $k \in \mathbb{Z}$ , the cubes of side length  $2^k$  form a partition of  $\mathbb{R}^n$ .

Let  $\mathcal{S}$  be a family of cubes and  $\eta \in (0, 1)$ . We say that  $\mathcal{S}$  is  $\eta$ -sparse, if, for each fixed  $Q \in \mathcal{S}$ , there exists a measurable subset  $E_Q \subset Q$ , such that  $|E_Q| \geq \eta|Q|$  and  $\{E_Q\}$  are pairwise disjoint. A family is called simply sparse if  $\eta = 1/2$ .

For constants  $\beta_1, \dots, \beta_m \in [0, \infty)$ , let  $\vec{\beta} = (\beta_1, \dots, \beta_m)$ . Associated with the sparse family  $\mathcal{S}$  and  $\vec{\beta}$ , we define the sparse operator  $\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}}}$  by

$$(2.1) \quad \mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x) = \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \|f_j\|_{L(\log L)^{\beta_j}, Q} \chi_Q(x),$$

with

$$\|f_j\|_{L(\log L)^{\beta_j}, Q} = \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|f(y)|}{\lambda} \log^{\beta_j} \left( 1 + \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.$$

For the case of  $\vec{\beta} = (0, \dots, 0)$ , we denote  $\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}}}$  by  $\mathcal{A}_{m; \mathcal{S}}$  for simplicity. Also, we denote  $\mathcal{A}_{1; \mathcal{S}, L(\log L)^{\beta}}(\mathcal{A}_1; \mathcal{S})$  by  $\mathcal{A}_{\mathcal{S}, L(\log L)^{\beta}}(\mathcal{A}_\mathcal{S})$ . For a weight  $u$ , let

$$\langle h \rangle_Q^u = \frac{1}{u(Q)} \int_Q h(y) u(y) dy,$$

and

$$(2.2) \quad \tilde{\mathcal{A}}_{m; \mathcal{S}}(f_1, \dots, f_m)(x) = \mathcal{A}_{m; \mathcal{S}}(f_1 \sigma_1, \dots, f_m \sigma_m)(x) = \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle f_j \rangle_Q^{\sigma_j} \langle \sigma_j \rangle_Q \chi_Q(x).$$

For a dyadic grid  $\mathcal{D}$ , and sparse family  $\mathcal{S} \subset \mathcal{D}$ , it was proved in [30] that for  $p_1, \dots, p_m \in (1, \infty)$ ,  $p \in (0, \infty)$  such that  $1/p = 1/p_1 + \dots + 1/p_m$ ,  $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$ , and  $\sigma_j = w_j^{-\frac{1}{p_j-1}}$  with  $j = 1, \dots, m$ ,

$$(2.3) \quad \|\tilde{\mathcal{A}}_{m; \mathcal{S}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, \sigma_j)},$$

and so

$$(2.4) \quad \|\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}.$$

**Theorem 2.1.** *Let  $p_1, \dots, p_m \in (1, \infty)$ ,  $p \in (0, \infty)$  such that  $1/p = 1/p_1 + \dots + 1/p_m$ , and  $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}(\mathbb{R}^{mn})$ . Set  $\sigma_i = w_i^{-1/(p_i-1)}$ . Let  $\mathcal{D}$  be a dyadic*

grid and  $\mathcal{S} \subset \mathcal{D}$  be a sparse family. Then for  $\beta_1, \dots, \beta_m \in [0, \infty)$ ,

$$(2.5) \quad \begin{aligned} & \|\mathcal{A}_{m;\mathcal{S},L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \\ & \lesssim [\vec{w}]_{A_p}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \prod_{j=1}^m [\sigma_j]_{A_\infty}^{\beta_j} \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}. \end{aligned}$$

*Proof.* We employ the ideas used in the proof of Theorem 3.2 in [30], in which Theorem 2.1 was proved for the case of  $\beta_1 = \beta_2 = 0$ , see also the proof of Theorem B in [1]. As it is well known,  $\vec{w} \in A_{\vec{p}}(\mathbb{R}^{mn})$  implies  $\sigma_j = w_j^{-\frac{1}{p_j-1}} \in A_{mp'_j}(\mathbb{R}^n)$  (see [27]). Also, it was pointed out in [25] that for the constant  $\theta_\sigma = 1 + \frac{1}{\tau_n[\sigma]_{A_\infty}}$  with  $\tau_n = 2^{11+n}$ ,

$$(2.6) \quad \left( \frac{1}{|Q|} \int_Q \sigma_j^{r_{\sigma_j}}(x) dx \right)^{\frac{1}{r_{\sigma_j}}} \leq 2 \frac{1}{|Q|} \int_Q \sigma_j(x) dx.$$

Let  $\varrho_j = (1 + p_j)/2$ . We can verify that

$$\|\sigma_j^{\frac{1}{\varrho_j}}\|_{L^{\varrho'_j}(\log L)^{\varrho'_j \beta_j}, Q} \lesssim \|\sigma_j\|_{L(\log L)^{\varrho'_j \beta_j}, Q}^{\frac{1}{\varrho_j}}.$$

Recall that

$$(2.7) \quad \|h\|_{L(\log L)^{\varrho}, Q} \lesssim \max\{1, \frac{1}{(\delta-1)^{\varrho}}\} \left( \frac{1}{|Q|} \int_Q |h(y)|^{\delta} dy \right)^{\frac{1}{\delta}}.$$

It then follows that

$$\begin{aligned} \|\sigma_j^{\frac{1}{\varrho_j}}\|_{L^{\varrho'_j}(\log L)^{\varrho'_j \beta_j}, Q} & \lesssim \frac{1}{(r_{\sigma_j} - 1)^{\beta_j}} \left( \frac{1}{|Q|} \int_Q \sigma_j^{r_{\sigma_j}}(y) dy \right)^{\frac{1}{\varrho'_j r_{\sigma_j}}} \\ & \lesssim [\sigma_j]_{A_\infty}^{\beta_j} \left( \frac{1}{|Q|} \int_Q \sigma_j(y) dy \right)^{\frac{1}{\varrho'_j}}. \end{aligned}$$

Applying the generalization of Hölder's inequality (see [37]), we deduce that

$$(2.8) \quad \begin{aligned} \|f_j \sigma_j\|_{L(\log L)^{\beta_j}, Q} & \lesssim \left( \frac{1}{|Q|} \int_Q |f_j|^{\varrho_j} \sigma_j \right)^{\frac{1}{\varrho_j}} \|\sigma_j^{\frac{1}{\varrho_j}}\|_{L^{\varrho'_j}(\log L)^{\varrho'_j \beta_j}, Q} \\ & \lesssim [\sigma_j]_{A_\infty}^{\beta_j} \left( \frac{1}{|Q|} \int_Q |f_j|^{\varrho_j} \sigma_j \right)^{\frac{1}{\varrho_j}} \left( \frac{1}{|Q|} \int_Q \sigma_j \right)^{\frac{1}{\varrho'_j}} \\ & = [\sigma_j]_{A_\infty}^{\beta_j} \left( \frac{1}{\sigma_j(Q)} \int_Q |f_j|^{\varrho_j} \sigma_j \right)^{\frac{1}{\varrho_j}} \frac{\sigma_j(Q)}{|Q|} \\ & \lesssim [\sigma_j]_{A_\infty}^{\beta_j} \langle M_{\sigma_j, \varrho_j}^{\mathcal{D}} f_j \rangle_Q^{\sigma_j} \langle \sigma_j \rangle_Q, \end{aligned}$$

here and in the following,  $M_{\sigma_j, \varrho_j}^{\mathcal{D}}$  is the maximal operator defined by

$$M_{\sigma_j, \varrho_j}^{\mathcal{D}} f_j(x) = \sup_{I \ni x, I \in \mathcal{D}} \left( \frac{1}{\sigma_j(I)} \int_I |f_j(y)|^{\varrho_j} \sigma_j(y) dy \right)^{\frac{1}{\varrho_j}}.$$

We then deduce that

$$\prod_{j=1}^m \|f_j \sigma_j\|_{L(\log L)^{\beta_j}, Q} \lesssim \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\beta_i} \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle M_{\sigma_j, \varrho_j}^{\mathcal{D}} f_j \rangle_Q^{\sigma_j} \langle \sigma_j \rangle_Q.$$



This, via the estimate (2.3) and the fact that  $M_{\sigma_j, \varrho_j}^{\mathcal{D}}$  is bounded on  $L^{p_j}(\mathbb{R}^n, \sigma_j)$  with bounds independent of  $\sigma_j$ , yields

$$\begin{aligned} & \left\| \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \|f_j \sigma_j\|_{L(\log L)^{\beta_j}, Q} \chi_Q \right\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \\ & \lesssim \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\beta_i} \left\| \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle M_{\sigma_j, \varrho_j} f_j \rangle_Q^{\sigma_j} \langle \sigma_j \rangle_Q \chi_Q \right\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \\ & \lesssim [\vec{w}]_{A_p}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} \prod_{i=1}^m [\sigma_i]_{A_\infty}^{\beta_i} \prod_{j=1}^m \|M_{\sigma_j, \varrho_j}^{\mathcal{D}} f_j\|_{L^{p_j}(\mathbb{R}^n, \sigma_j)} \end{aligned}$$

and then completes the proof of Theorem 2.1.  $\square$

For locally integrable functions  $b_1, \dots, b_m$  and a sparse family  $\mathcal{S}$ , let

$$(2.9) \quad \mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m)(x) = \sum_{Q \in \mathcal{S}} \left( \sum_{i=1}^m |b_i(x) - \langle b_i \rangle_Q| \right) \prod_{j=1}^m \langle f_j \rangle_Q \chi_Q(x).$$

**Theorem 2.2.** *Let  $p_1, \dots, p_m \in (1, \infty)$ ,  $p \in (0, \infty)$  such that  $1/p = 1/p_1 + \dots + 1/p_m$ , and  $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{p}}(\mathbb{R}^{mn})$ . Let  $\mathcal{D}$  be a dyadic grid and  $\mathcal{S} \subset \mathcal{D}$  be a sparse family,  $b_i \in \text{Osc}_{\exp L^{s_i}}(\mathbb{R}^n)$  ( $s_i \in [1, \infty)$ ) with  $\sum_{i=1}^m \|b_i\|_{\text{Osc}_{\exp L^{s_i}}(\mathbb{R}^n)} = 1$ . Then*

$$(2.10) \quad \|\mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_p}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p}\}} [\nu_{\vec{w}}]_{A_\infty}^{\frac{1}{s_*}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}.$$

*Proof.* We first consider the case of  $p \in (0, 1]$ . Write

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathcal{A}_{m; \mathcal{S}, \vec{b}} \vec{f}(x))^p \nu_{\vec{w}}(x) dx & \leq \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q^p \sum_{i=1}^m \int_{\mathbb{R}^n} |b_i(x) - \langle b_i \rangle_Q|^p \nu_{\vec{w}}(x) dx \\ & \leq \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q^p \sum_{i=1}^m |Q| \|\nu_{\vec{w}}\|_{L(\log L)^{\frac{p}{s_i}}, Q} \\ & \leq [\nu_{\vec{w}}]_{A_\infty}^{\frac{p}{s_*}} \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q^p \nu_{\vec{w}}(Q), \end{aligned}$$

where in the last inequality, we have invoked the estimates (2.7) and (2.6) for  $\nu_{\vec{w}}$ . It was proved in [30, pp. 757-758] that

$$\sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q^p \nu_{\vec{w}}(Q) \lesssim [\vec{w}]_{A_p}^{\max\{p'_1, \dots, p'_m\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}^p.$$

The inequality (2.10) then follows in this case.

To consider the case of  $p \in (1, \infty)$ , let  $\varrho = \frac{1+p'}{2}$  with  $p' = \frac{p}{p-1}$ . Observe that by (2.7),

$$\begin{aligned} \|g \nu_{\vec{w}}\|_{L(\log L)^{\frac{1}{s_*}}, Q} & \lesssim \left( \frac{1}{|Q|} \int_Q |g(x)|^\varrho \nu_{\vec{w}}(x) dx \right)^{\frac{1}{\varrho}} \|\nu_{\vec{w}}^{\frac{1}{\varrho}}\|_{L^{\varrho'}(\log L)^{\frac{\varrho'}{s_*}}, Q} \\ & \lesssim [w]_{A_\infty}^{\frac{1}{s_*}} \left( \frac{1}{\nu_{\vec{w}}(Q)} \int_Q |g(x)|^\varrho \nu_{\vec{w}}(x) dx \right)^{\frac{1}{\varrho}} \frac{\nu_{\vec{w}}(Q)}{|Q|}. \end{aligned}$$

Therefore, by the generalization of Hölder's inequality (see [37]),

$$\begin{aligned}
& \int_{\mathbb{R}^n} \mathcal{A}_{m, \mathcal{S}, \bar{b}}(f_1, \dots, f_m)(x) g(x) \nu_{\bar{w}}(x) dx \\
&= \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q \sum_{i=1}^m \int_{\mathbb{R}^n} |b_i(x) - \langle b_i \rangle_Q| g(x) \nu_{\bar{w}}(x) dx \\
&\leq \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q |Q| \|g \nu_{\bar{w}}\|_{L(\log L)^{\frac{1}{s_*}}, Q} \\
&\leq [\nu_{\bar{w}}]_{A_\infty}^{\frac{1}{s_*}} \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q \inf_{x \in Q} M_{\nu_{\bar{w}}, \varrho}^{\mathcal{D}} g(x) \nu_{\bar{w}}(Q) \\
&\leq [\nu_{\bar{w}}]_{A_\infty}^{\frac{1}{s_*}} \|\mathcal{A}_{\mathcal{S}}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^n, \nu_{\bar{w}})} \|M_{\nu_{\bar{w}}, \varrho}^{\mathcal{D}} g\|_{L^{p'}(\mathbb{R}^n, \nu_{\bar{w}})}.
\end{aligned}$$

Our desired conclusion then follows from (2.4) and the  $L^{p'}(\mathbb{R}^n, \nu_{\bar{w}})$  boundedness of  $M_{\nu_{\bar{w}}, \varrho}^{\mathcal{D}}$ .  $\square$

### 3. PROOF OF THEOREMS 1.5 AND 1.8

Let  $T$  be an  $m$ -sublinear operator. Associated with  $T$ , let

$$\mathcal{M}_T(f_1, \dots, f_m)(x) = \sup_{Q \ni x} \|T(f_1, \dots, f_m)(\xi) - T(f_1 \chi_{3Q}, \dots, f_m \chi_{3Q})(\xi)\|_{L^\infty(Q)}.$$

Following the argument in [26], we have

**Lemma 3.1.** *Let  $q_1, \dots, q_m \in (1, \infty)$ ,  $q \in (1/m, \infty)$  such that  $1/q = 1/q_1 + \dots + 1/q_m$ ,  $T$  be an  $m$ -sublinear operator which is bounded from  $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$  to  $L^{\frac{1}{m}, \infty}(l^q; \mathbb{R}^n)$ . Then for any cube  $Q_0$  and a. e.  $x \in Q_0$ , we have that*

$$\begin{aligned}
\| \{T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x)\} \|_{l^q} &\leq C_1 \prod_{j=1}^m \| \{f_j^k(x)\} \|_{l^{q_j}} \\
&\quad + \| \{ \mathcal{M}_T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x) \} \|_{l^q},
\end{aligned}$$

provided that  $\| \{f_1^k\} \|_{l^{q_1}}, \dots, \| \{f_m^k\} \|_{l^{q_m}} \in L_{loc}^1(\mathbb{R}^n)$ .

*Proof.* We follow the line in [28]. Let  $x \in \text{int} Q_0$  be a point of approximation continuity of  $\| \{T(f_1 \chi_{3Q_0}, \dots, f_m \chi_{3Q_0})\} \|_{l^q}$ . For  $r, \epsilon > 0$ , the set

$$\begin{aligned}
E_r(x) &= \{y \in B(x, r) : \| \{T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x)\} \|_{l^q} \\
&\quad - \| \{T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(y)\} \|_{l^q} \| < \epsilon \}
\end{aligned}$$

satisfies that  $\lim_{r \rightarrow 0} \frac{|E_r(x)|}{|B(x, r)|} = 1$ . Denote by  $Q(x, r)$  the smallest cube centered at  $x$  and containing  $B(x, r)$ . Let  $r > 0$  small enough such that  $Q(x, r) \subset Q_0$ . Then for  $y \in E_r(x)$ ,

$$\begin{aligned}
\| \{T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x)\} \|_{l^q} &< \| \{T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(y)\} \|_{l^q} + \epsilon \\
&\leq \| \{T(f_1^k \chi_{3Q(x, r)}, \dots, f_m^k \chi_{3Q(x, r)})(y)\} \|_{l^q} \\
&\quad + \| \{ \mathcal{M}_T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x) \} \|_{l^q} + \epsilon.
\end{aligned}$$

Thus, for  $\varsigma \in (0, 1/m)$ ,

$$\begin{aligned} & \left\| \{T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x)\} \right\|_{l^q} \\ & \leq \left( \frac{1}{|E_s(x)|} \int_{E_s(x)} \left\| \{T(f_1^k \chi_{3Q(x,r)}, \dots, f_m^k \chi_{3Q(x,r)})(y)\} \right\|_{l^q}^\varsigma dy \right)^{\frac{1}{\varsigma}} \\ & \quad + \left\| \{\mathcal{M}_T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x)\} \right\|_{l^q} + \epsilon \\ & \leq C \prod_{j=1}^m \langle \|f_j^k\|_{l^{q_j}} \rangle_{Q(x,r)} + \left\| \{\mathcal{M}_T(f_1^k \chi_{3Q_0}, \dots, f_m^k \chi_{3Q_0})(x)\} \right\|_{l^q} + \epsilon, \end{aligned}$$

since  $T$  is bounded from  $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$  to  $L^{\frac{1}{m}, \infty}(l^q; \mathbb{R}^n)$ . Taking  $r \rightarrow 0+$  then leads to the conclusion (i).  $\square$

**Lemma 3.2.** *Let  $\tau \in (0, 1)$  and  $M_\tau$  be the maximal operator defined by*

$$M_\tau f(x) = (M(|f|^\tau)(x))^{\frac{1}{\tau}}.$$

*Then for any  $p \in (\tau, \infty)$  and  $u \in A_{p/\tau}(\mathbb{R}^n)$*

$$u(\{x \in \mathbb{R}^n : \|\{M_\tau f_k(x)\}\|_{l^q} > \lambda\}) \lesssim_{u,p} \lambda^{-p} \sup_{t \geq C\lambda} t^p u(\{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{l^q} > t\}).$$

*Proof.* For each fixed  $\lambda > 0$ , decompose  $f_k$  as

$$f_k(y) = f_k(y) \chi_{\{\|f_k(y)\|_{l^q} \leq \lambda\}}(y) + f_k(y) \chi_{\{\|f_k(y)\|_{l^q} > \lambda\}}(y) := f_k^1(y) + f_k^2(y).$$

It then follows that

$$u(\{x \in \mathbb{R}^n : \|\{M_\tau f_k(x)\}\|_{l^q} > 2^{\frac{1}{\tau}} \lambda\}) \leq u(\{x \in \mathbb{R}^n : \|\{M(|f_k^2|^\tau)(x)\}\|_{l^{\frac{q}{\tau}}} > \lambda^\tau\}).$$

Recall that  $u \in A_{p/\tau}$  implies that  $u \in A_{\frac{p-\epsilon}{\tau}}(\mathbb{R}^n)$  for some  $\epsilon \in (0, p-\tau)$ , and that  $M$  is bounded on  $L^{\frac{p-\epsilon}{\tau}}(l^q; \mathbb{R}^n, u)$  (see [15]). Therefore,

$$\begin{aligned} & u(\{x \in \mathbb{R}^n : \|\{M(|f_k^2|^\tau)(x)\}\|_{l^{\frac{q}{\tau}}} > \lambda^\tau\}) \\ & \lesssim \lambda^{-p+\epsilon} \int_{\mathbb{R}^n} \|\{f_k^2(x)\}\|_{l^q}^{p-\epsilon} u(x) dx \\ & \lesssim u(\{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{l^q} > \lambda\}) \\ & \quad + \lambda^{-p+\epsilon} \int_{\lambda}^{\infty} u(\{x \in \mathbb{R}^n : \|\{f_k^2(x)\}\|_{l^q} > t\}) t^{p-\epsilon-1} dt \\ & \lesssim \lambda^{-p} \sup_{t \geq \lambda} t^p u(\{x \in \mathbb{R}^n : \|\{f_k(x)\}\|_{l^q} > t\}). \end{aligned}$$

This yields our desired conclusion.  $\square$

**Lemma 3.3.** *Let  $q_1, \dots, q_m \in (1, \infty)$ ,  $q \in (1/m, \infty)$  such that  $1/q = 1/q_1 + \dots + 1/q_m$ . Under the hypothesis of Theorem 1.5, the operator  $\mathcal{M}_T$  is bounded from  $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$  to  $L^{\frac{1}{m}, \infty}(l^q; \mathbb{R}^n)$ .*

*Proof.* For simplicity, we only consider the bilinear case, namely,  $m = 2$ . For  $\epsilon > 0$ , let

$$T^\epsilon(f_1, f_2)(x) = \int_{\max_j |x-y_j| > \epsilon} K(x; y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.$$

We claim that for each  $\tau \in (0, 1/2)$ ,

$$(3.1) \quad \sup_{\epsilon > 0} |T^\epsilon(f_1, f_2)(x)| \lesssim M_\tau(T(f_1, f_2))(x) + M f_1(x) M f_2(x).$$

To prove this, we will employ the ideas used in [14, 17]. let

$$G^\epsilon(f_1, f_2)(x, z) = \int_{\min_j |x-y_j| > \epsilon} K(z; y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.$$

For functions  $f_1, \dots, f_m$ , set

$$f_j^{(0)}(y) = f_j(y) \chi_{B(x, \epsilon)}(y).$$

Let

$$A_\epsilon(f_1, f_2)(x) = \int_{\substack{\max_j |x-y_j| > \epsilon, \\ \min_j |x-y_j| \leq \epsilon}} |K(x; y_1, y_2)| |f_1(y_1) f_2(y_2)| dy_1 dy_2,$$

and

$$E_\epsilon(f_1, f_2)(x, z) = \int_{\substack{\max_j |x-y_j| > \epsilon, \\ \min_j |x-y_j| \leq \epsilon}} |K(z; y_1, y_2)| |f_1(y_1) f_2(y_2)| dy_1 dy_2.$$

By the size condition, it is easy to verify that

$$A_\epsilon(f_1, f_2)(x) \lesssim M f_1(x) M f_2(x).$$

Also, for  $z \in B(x, \epsilon/8)$ , we have

$$E_\epsilon(f_1, f_2)(x, z) \lesssim M f_1(x) M f_2(x).$$

It then follows from (1.11) that for  $z \in B(x, \epsilon/8)$ ,

$$\begin{aligned} & |T^\epsilon(f_1, f_2)(x) - G^\epsilon(f_1, f_2)(x, z)| \\ & \lesssim A_\epsilon(f_1, f_2)(x) + E_\epsilon(f_1, f_2)(x, z) \\ & \quad + \int_{\min_i |x-y_i| > \epsilon} |K(x; y_1, y_2) - K(z; y_1, y_2)| f_1(y_1) f_2(y_2) dy_1 dy_2 \\ & \lesssim M f_1(x) M f_2(x). \end{aligned}$$

Observe that for  $z \in B(x, \epsilon/8)$ ,

$$\begin{aligned} |G^\epsilon(f_1, f_2)(x, z)| & \leq \left| \int_{\max_j |z-y_j| > \epsilon} K(z; y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \right| \\ & \quad + \int_{\frac{\epsilon}{2} \leq \max_j |x-y_j| \leq 2\epsilon} |K(z; y_1, y_2)| |f_1(y_1) f_2(y_2)| dy_1 dy_2 \\ & \leq |T(f_1, f_2)(z)| + |T(f_1^{(0)}, f_2^{(0)})(z)| + M f_1(x) M f_2(x). \end{aligned}$$

Therefore, for any  $z \in B(x, \epsilon/8)$ ,

$$|T^\epsilon(f_1, f_2)(x)| \leq |T(f_1, f_2)(z)| + |T(f_1^{(0)}, f_2^{(0)})(z)| + \prod_{i=1}^2 M f_i(x).$$

This, together with the fact that  $T$  is bounded from  $L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n)$  to  $L^{1/m, \infty}(\mathbb{R}^n)$ , leads to (3.1).

Now let

$$T_\epsilon(f_1, f_2)(x) = \int_{\min_j |x-y_j| > \epsilon} K(x; y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.$$

By the size condition (1.7), we see that

$$|T^\epsilon(f_1, f_2)(x) - T_\epsilon(f_1, f_2)(x)| \lesssim M f_1(x) M f_2(x).$$

and so

$$\sup_{\epsilon > 0} |T_\epsilon(f_1, f_2)(x)| \lesssim M_\tau(T(f_1, f_2))(x) + Mf_1(x)Mf_2(x).$$

Let  $Q \subset \mathbb{R}^n$  be a cube and  $x, \xi \in Q$ . Denote by  $B_x$  the ball centered at  $x$  and having diameter  $10\text{diam} Q$ . Then  $3Q \subset B_x$ . As in [28], we write

$$\begin{aligned} & |T(f_1\chi_{\mathbb{R}^n \setminus 3Q}, f_2\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \\ & \leq |T(f_1\chi_{\mathbb{R}^n \setminus B_x}, f_2\chi_{\mathbb{R}^n \setminus B_x})(\xi) - T(f_1\chi_{\mathbb{R}^n \setminus B_x}, f_2\chi_{\mathbb{R}^n \setminus B_x})(x)| + \sup_{\epsilon > 0} |T_\epsilon(f_1, f_2)(x)| \\ & \quad + |T(f_1\chi_{\mathbb{R}^n \setminus B_x}, f_2\chi_{B_x \setminus 3Q})(\xi)| + |T(f_1\chi_{B_x \setminus 3Q}, f_2\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \end{aligned}$$

It follows from the regularity condition (1.11) that

$$|T(f_1\chi_{\mathbb{R}^n \setminus B_x}, f_2\chi_{\mathbb{R}^n \setminus B_x})(\xi) - T(f_1\chi_{\mathbb{R}^n \setminus B_x}, f_2\chi_{\mathbb{R}^n \setminus B_x})(x)| \lesssim \prod_{i=1}^2 Mf_i(x).$$

On the other hand, by the size condition (1.7), we have

$$\begin{aligned} |T(f_1\chi_{B_x \setminus 3Q}, f_2\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| & \lesssim \int_{B_x} |f_1(y_1)| dy_1 \int_{\mathbb{R}^n \setminus 3Q} \frac{|f_2(y_2)|}{(|x - y_2| + \text{diam} Q)^{2n}} dy_2 \\ & \lesssim Mf_1(x)Mf_2(x). \end{aligned}$$

Similarly,

$$|T(f_1\chi_{\mathbb{R}^n \setminus B_x}, f_2\chi_{B_x \setminus 3Q})(\xi)| \lesssim Mf_1(x)Mf_2(x),$$

and

$$|T(f_1\chi_{\mathbb{R}^n \setminus 3Q}, f_2\chi_{3Q})(\xi) + T(f_1\chi_{3Q}, f_2\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \lesssim Mf_1(x)Mf_2(x).$$

Combining the estimates above leads to that

$$(3.2) \quad \mathcal{M}_T(f_1, f_2)(x) \lesssim M_\tau(T(f_1, f_2))(x) + \prod_{i=1}^2 Mf_i(x).$$

Recall that  $T$  is bounded from  $L^1(l^{q_1}; \mathbb{R}^n) \times L^1(l^{q_2}; \mathbb{R}^n)$  to  $L^{\frac{1}{2}, \infty}(l^q; \mathbb{R}^n)$  (see [21]), and  $M$  is bounded from  $L^1(l^{q_j}; \mathbb{R}^n)$  to  $L^{1, \infty}(l^{q_j}; \mathbb{R}^n)$ . Now we choose  $\tau \in (0, 1/2)$  in (3.2), our desired conclusion now follows from (3.2) and Lemma 3.2 immediately.  $\square$

**Theorem 3.4.** *Let  $q_1, \dots, q_m \in (1, \infty)$  and  $q \in (1/m, \infty)$  with  $1/q = 1/q_1 + \dots + 1/q_m$ . Suppose that both the operators  $T$  and  $\mathcal{M}_T$  are bounded from  $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$  to  $L^{1/m, \infty}(l^q; \mathbb{R}^n)$ . Then for  $N \in \mathbb{N}$  and bounded functions  $\{f_1^k\}_{1 \leq k \leq N}, \dots, \{f_m^k\}_{1 \leq k \leq N}$  with compact supports, there exists a  $\frac{1}{2} \frac{1}{3^n}$ -sparse of family  $\mathcal{S}$  such that for a. e.  $x \in \mathbb{R}^n$ ,*

$$(3.3) \quad \|\{T(f_1^k, \dots, f_m^k)(x)\}\|_{l^q} \lesssim \mathcal{A}_{m; \mathcal{S}}(\|\{f_1^k\}\|_{l^{q_1}}, \dots, \|\{f_m^k\}\|_{l^{q_m}})(x).$$

*Proof.* Again, we only consider the case  $m = 2$ . We follow the argument used in [28]. At first, we claim that for each cube  $Q_0 \subset \mathbb{R}^n$ , there exist pairwise disjoint cubes  $\{P_j\} \subset \mathcal{D}(Q_0)$ , such that  $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$  and a. e.  $x \in Q_0$ ,

$$\begin{aligned} (3.4) \quad & \|\{T(f_1^k\chi_{3Q_0}, f_2^k\chi_{3Q_0})(x)\}\|_{l^q\chi_{Q_0}}(x) \\ & \leq C \prod_{i=1}^2 \langle \|\{f_i^k\}\|_{l^{q_i}} \rangle_{3Q_0} + \sum_j \|\{T(f_1^k\chi_{3P_j}, f_2^k\chi_{3P_j})(x)\}\|_{l^q\chi_{P_j}}(x). \end{aligned}$$

To prove this, let  $C_2 > 0$  which will be chosen later and

$$\begin{aligned} E &= \{x \in Q_0 : \|\{f_1^k(x)\}\|_{l^{q_1}} \|\{f_2^k(x)\}\|_{l^{q_2}} > \prod_{i=1}^2 \langle \|\{f_i^k\}\|_{l^{q_i}} \rangle_{3Q_0}\} \\ &\cup \{x \in Q_0 : \|\{\mathcal{M}_T(f_1^k \chi_{3Q_0}, f_2^k \chi_{3Q_0})(x)\}\|_{l^q} > C_2 \langle \prod_{i=1}^2 \langle \|\{f_i^k\}\|_{l^{q_i}} \rangle_{3Q_0} \rangle\}. \end{aligned}$$

If we choose  $C_2$  large enough, we then know from Lemma 3.3 that  $|E| \leq \frac{1}{2^{n+2}}|Q_0|$ . Now applying the Calderón-Zygmund decomposition to  $\chi_E$  on  $Q_0$  at level  $\frac{1}{2^{n+1}}$ , we then obtain a family of pairwise disjoint cubes  $\{P_j\}$  such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|,$$

and  $|E \setminus \cup_j P_j| = 0$ . It then follows that  $\sum_j |P_j| \leq \frac{1}{2}|E|$ , and  $P_j \cap E^c \neq \emptyset$ . Therefore,

$$\begin{aligned} (3.5) \quad &\left\| \left\{ T(f_1^k \chi_{3Q_0 \setminus 3P_j}, f_2^k \chi_{3Q_0 \setminus 3P_j})(\xi) \right\} \right\|_{l^q} + \left\| \left\{ T(f_1^k \chi_{3Q_0 \setminus 3P_j}, f_2^k \chi_{3P_j})(\xi) \right\} \right\|_{l^q} \\ &+ \left\| \left\{ T(f_1^k \chi_{3P_j}, f_2^k \chi_{3Q_0 \setminus 3P_j})(\xi) \right\} \right\|_{l^q} \Big\|_{L^\infty(P_j)} \leq C_2 \prod_{i=1}^2 \langle \|\{f_i^k\}\|_{l^{q_i}} \rangle_{3Q_0}. \end{aligned}$$

Note that

$$\begin{aligned} (3.6) \quad &\left\| \left\{ T(f_1^k \chi_{3Q_0}, f_2^k \chi_{3Q_0})(x) \right\} \right\|_{l^q} \chi_{Q_0}(x) \\ &\leq \left\| \left\{ T(f_1^k \chi_{3Q_0}, f_2^k \chi_{3Q_0})(x) \right\} \right\|_{l^q} \chi_{Q_0 \setminus \cup_j P_j}(x) \\ &\quad + \sum_j \left\| \left\{ T(f_1^k \chi_{3P_j}, f_2^k \chi_{3P_j})(x) \right\} \right\|_{l^q} \chi_{P_j}(x) + \sum_j D_j(x) \chi_{P_j}(x), \end{aligned}$$

with

$$\begin{aligned} D_j(x) &= \left\| \left\{ T(f_1^k \chi_{3Q_0 \setminus 3P_j}, f_2^k \chi_{3Q_0 \setminus 3P_j})(x) \right\} \right\|_{l^q} + \left\| \left\{ T(f_1^k \chi_{3Q_0 \setminus 3P_j}, f_2^k \chi_{3P_j})(x) \right\} \right\|_{l^q} \\ &\quad + \left\| \left\{ T(f_1^k \chi_{3P_j}, f_2^k \chi_{3Q_0 \setminus 3P_j})(x) \right\} \right\|_{l^q}. \end{aligned}$$

(3.4) now follows from (3.5), (3.6) and Lemma 3.1.

We can now conclude the proof of Theorem 3.4. As it was proved in [26], the last estimate shows that there exists a  $\frac{1}{2}$ -sparse family  $\mathcal{F} \subset \mathcal{D}(Q_0)$ , such that for a. e.  $x \in Q_0$ ,

$$\left\| \left\{ T(f_1^k \chi_{3Q_0}, f_2^k \chi_{3Q_0})(x) \right\} \right\|_{l^q} \chi_{Q_0}(x) \lesssim \sum_{Q \in \mathcal{F}} \prod_{i=1}^2 \langle \|\{f_i^k\}\|_{l^{q_i}} \rangle_{3Q} \chi_Q(x).$$

Recalling that  $\{f_1^k\}_{1 \leq k \leq N}$ ,  $\{f_2^k\}_{1 \leq k \leq N}$  are functions in  $L^1(\mathbb{R}^n)$  with compact supports, we can take now a partition of  $\mathbb{R}^n$  by cubes  $Q_j$  such that  $\cup_{k=1}^N \cup_{i=1}^2 \text{supp } f_i^k \subset 3Q_j$  for each  $j$  and obtain a  $\frac{1}{2}$ -sparse family  $\mathcal{F}_j \subset \mathcal{D}(Q_j)$  such that for a. e.  $x \in Q_j$ ,

$$\left\| \left\{ T(f_1^k \chi_{3Q_j}, f_2^k \chi_{3Q_j})(x) \right\} \right\|_{l^q} \chi_{Q_j}(x) \lesssim \sum_{Q \in \mathcal{F}_j} \prod_{i=1}^2 \langle \|\{f_i^k\}\|_{l^{q_i}} \rangle_{3Q} \chi_Q(x).$$

Setting  $\mathcal{S} = \{3Q : Q \in \cup_j \mathcal{F}_j\}$ , we see that (3.3) holds true for  $\mathcal{S}$  and a. e.  $x \in \mathbb{R}^n$ .  $\square$

Similar to the proof of Theorem 3.4, by mimicking the proof of Theorem 1.1 in [28], we can prove

**Theorem 3.5.** *Let  $q_1, \dots, q_m \in (1, \infty)$  and  $q \in (1/m, \infty)$  with  $1/q = 1/q_1 + \dots + 1/q_m$ ,  $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ . Suppose that both the operators  $T$  and  $\mathcal{M}_T$  are bounded from  $L^1(l^{q_1}; \mathbb{R}^n) \times \dots \times L^1(l^{q_m}; \mathbb{R}^n)$  to  $L^{1/m, \infty}(l^q; \mathbb{R}^n)$ . Then for  $N \in \mathbb{N}$  and bounded functions  $\{f_1^k\}_{1 \leq k \leq N}, \dots, \{f_m^k\}_{1 \leq k \leq N}$  with compact supports, there exists a  $\frac{1}{2} \frac{1}{3^n}$ -sparse of family  $\mathcal{S}$  such that for a. e.  $x \in \mathbb{R}^n$ ,*

$$\begin{aligned} \|\{[b, T]_i(f_1^k, \dots, f_m^k)(x)\}\|_{l^q} &\lesssim \sum_{Q \in \mathcal{S}} \langle |b - \langle b \rangle_Q| \|\{f_i^k\}\|_{l^{q_i}} \rangle_Q \prod_{j \neq i} \langle \|\{f_j^k\}\|_{l^{q_j}} \rangle_Q \chi_Q(x) \\ &\quad + \sum_{Q \in \mathcal{S}} |b(x) - \langle b \rangle_Q| \prod_{j=1}^m \langle \|\{f_j^k\}\|_{l^{q_j}} \rangle_Q \chi_Q(x). \end{aligned}$$

We are now ready to prove Theorem 1.5 and Theorem 1.8.

*Proof of Theorem 1.5.* Obviously, it suffices to consider the case that  $\{f_1^k\}, \dots, \{f_m^k\}$  are finite sequences. By the well known one-third trick (see [24, Lemma 2.5]), we know that if  $\mathcal{S}$  is a sparse family, then there exist general dyadic grids  $\mathcal{D}_1, \dots, \mathcal{D}_{3^n}$ , and sparse families  $\mathcal{S}_i \subset \mathcal{D}_i$ , with  $i = 1, \dots, 3^n$ , such that

$$\mathcal{A}_{m; \mathcal{S}, L(\log L)^\beta}(f_1, \dots, f_m)(x) \lesssim_n \sum_{i=1}^{3^n} \mathcal{A}_{m; \mathcal{S}_i, L(\log L)^\beta}(f_1, \dots, f_m)(x).$$

Thus, Theorem 1.5 follows from Theorem 3.4, Lemma 3.3 and the estimate (2.4) directly.  $\square$

*Proof of Theorem 1.8.* By the generalization of Hölder's inequality (see [37]), we know that

$$\langle |b_i(x) - \langle b_i \rangle_Q| \|\{f_i^k\}\|_{l^{q_i}} \rangle_Q \lesssim \|\|\{f_i^k\}\|_{l^{q_i}}\|_{L(\log L)^{\frac{1}{q_i}}, Q}.$$

For  $N \in \mathbb{N}$  and bounded functions  $\{f_1^k\}_{1 \leq k \leq N}, \dots, \{f_m^k\}_{1 \leq k \leq N}$  with compact supports, we know from Theorem 3.5 that there exists a  $\frac{1}{2} \frac{1}{3^n}$ -sparse of family  $\mathcal{S}$  such that for a. e.  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} (3.7) \quad \|\{T_b(f_1^k, \dots, f_m^k)(x)\}\|_{l^q} &\lesssim \sum_{i=1}^m \mathcal{A}_{m; \mathcal{S}, L(\log L)^{\beta_i}}(\|\{f_1^k\}\|_{l^{q_1}}, \dots, \|\{f_m^k\}\|_{l^{q_m}})(x) \\ &\quad + \mathcal{A}_{m; \mathcal{S}, \vec{\beta}}(\|\{f_1^k\}\|_{l^{q_1}}, \dots, \|\{f_m^k\}\|_{l^{q_m}})(x), \end{aligned}$$

with  $\vec{\beta}_1 = (\frac{1}{s_1}, 0, \dots, 0), \dots, \vec{\beta}_m = (0, \dots, 0, \frac{1}{s_m})$ . As in the proof of Lemma 3.3, Theorem 1.5, Theorem 1.8 follows from Theorem 2.1 and Theorem 2.2. We omit the details for brevity.

#### 4. PROOF OF THEOREM 1.9

For  $\beta_1, \dots, \beta_m \in [0, \infty)$ , let  $\mathcal{M}_{L(\log L)^{\vec{\beta}}}$  be the maximal operator defined by

$$\mathcal{M}_{L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x) = \sup_{Q \ni x} \prod_{j=1}^m \|f_j\|_{L(\log L)^{\beta_j}, Q}.$$

For the case of  $\vec{\beta} = (0, \dots, 0)$ , we denote  $\mathcal{M}_{L(\log L)^{\vec{\beta}}}$  by  $\mathcal{M}$ . As in [27] and [33], we can prove that

**Lemma 4.1.** *Let  $\beta_1, \dots, \beta_m \in [0, \infty)$ ,  $|\beta| = \beta_1 + \dots + \beta_m$  and  $\vec{w} = (w_1, \dots, w_m) \in A_{1, \dots, 1}(\mathbb{R}^{mn})$ . Then for each  $\lambda > 0$ ,*

$$\begin{aligned} & \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{M}_{L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x) > \lambda\}) \\ & \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \frac{|f_j(x)|}{\lambda^{\frac{1}{m}}} \log^{|\beta|} \left( 1 + \frac{|f_j(x)|}{\lambda^{\frac{1}{m}}} \right) w_j(x) dx \right)^{\frac{1}{m}}. \end{aligned}$$

The following conclusion was established by Lerner et al. in [28].

**Lemma 4.2.** *Let  $\beta \in [0, \infty)$  and  $\mathcal{S}$  be a sparse family of cubes. Then for each fixed  $\lambda > 0$ ,*

$$|\{x \in \mathbb{R}^n : \mathcal{A}_{\mathcal{S}, L(\log L)^{\beta}} f(x) > \lambda\}| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\beta} \left( 1 + \frac{|f(x)|}{\lambda} \right) dx,$$

and for  $b \in \text{BMO}(\mathbb{R}^n)$ ,

$$|\{x \in \mathbb{R}^n : \mathcal{A}_{\mathcal{S}, b} f(x) > \lambda\}| \lesssim \lambda^{-1} \int_{\mathbb{R}^n} |f(x)| dx.$$

**Lemma 4.3.** *Let  $\varrho \in [0, \infty)$  and  $\delta \in (0, 1)$ ,  $T$  be a sublinear operator which satisfies the weak type estimate that*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\varrho} \left( 1 + \frac{|f(x)|}{\lambda} \right) dx.$$

Then for any cube  $I$  and appropriate function  $f$  with  $\text{supp } f \subset I$ ,

$$(4.1) \quad \left( \frac{1}{|I|} \int_I |Tf(x)|^{\delta} dx \right)^{\frac{1}{\delta}} \lesssim \|f\|_{L(\log L)^{\varrho}, I}.$$

For the proof of Lemma 4.3, see [20].

**Lemma 4.4.** *Let  $m \geq 2$  be an integer,  $\mathcal{D}$  be a dyadic grid and  $\mathcal{S} \subset \mathcal{D}$  be a finite sparse family. Then for each fixed  $I \in \mathcal{D}$  and  $\delta \in (0, \frac{1}{m})$ .*

$$(4.2) \quad \inf_{c \in \mathbb{C}} \left( \frac{1}{|I|} \int_I |\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x) - c|^{\delta} dx \right)^{\frac{1}{\delta}} \lesssim_{\delta} \prod_{j=1}^m \|f_j\|_{L(\log L)^{\beta_j}, I};$$

and for  $b_i \in \text{Osc}_{\exp L^{s_i}}(\mathbb{R}^n)$  ( $s_i \in [1, \infty)$ ,  $i = 1, \dots, m$ ),  $\gamma \in (\delta, \frac{1}{m})$ ,

$$(4.3) \quad \begin{aligned} & \inf_{c \in \mathbb{C}} \left( \frac{1}{|I|} \int_I |\mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m)(x) - c|^{\delta} dx \right)^{\frac{1}{\delta}} \\ & \lesssim_{\delta} \inf_{y \in I} M_{\gamma}(\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m))(y) + \prod_{j=1}^m \langle |f_j| \rangle_I. \end{aligned}$$

*Proof.* Without loss of generality, we may assume that the functions  $f_1, \dots, f_m$  are nonnegative. Let  $c_0 = \sum_{Q \supset I} \prod_{j=1}^m \|f_j\|_{L(\log L)^{\beta_j}, Q}$ . As in [10], it follows that

$$\begin{aligned} & \int_I |\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}}}(f_1, \dots, f_m)(x) - c_0|^{\delta} dx \\ & \lesssim \int_I \left| \sum_{Q \in \mathcal{S}, Q \supset I} \|f_j\|_{L(\log L)^{\beta_j}, Q} \chi_Q(x) \right|^{\delta} dx \\ & \lesssim \int_I |\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}}}(f_1 \chi_I, \dots, f_m \chi_I)(x)|^{\delta} dx. \end{aligned}$$



On the other hand, by Lemma 4.2 and Lemma 4.3, we know that

$$\left( \frac{1}{|I|} \int_I \left| \mathcal{A}_{\mathcal{S}, L(\log L)^{\beta_j}}(f_j \chi_I)(x) \right|^{m\delta} dx \right)^{\frac{1}{m\delta}} \lesssim \|f_j\|_{L(\log L)^{\beta_j}, I}.$$

This, together with the fact that

$$\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\beta}}(f_1, \dots, f_m)(x) \lesssim \prod_{j=1}^m \mathcal{A}_{\mathcal{S}, L(\log L)^{\beta_j}} f_j(x).$$

and Hölder's inequality, leads to that

$$\begin{aligned} & \left( \frac{1}{|I|} \int_I \left| \mathcal{A}_{m; \mathcal{S}, L(\log L)^{\beta}}(f_1 \chi_I, \dots, f_m \chi_I)(x) \right|^{\delta} dx \right)^{\frac{1}{\delta}} \\ & \lesssim \prod_{j=1}^m \left( \frac{1}{|I|} \int_I \left| \mathcal{A}_{\mathcal{S}, L(\log L)^{\beta_j}}(f_j \chi_I)(x) \right|^{m\delta} dx \right)^{\frac{1}{m\delta}} \\ & \lesssim \prod_{j=1}^m \|f_j\|_{L(\log L)^{\beta_j}, I}. \end{aligned}$$

To prove (4.2), we first observe that, for each constant  $c \in \mathbb{C}$  and a cube  $I \subset \mathcal{D}$ ,

$$\begin{aligned} & |\mathcal{A}_{m; \mathcal{S}, \bar{b}}(f_1, \dots, f_m)(x) - c| \\ & \leq \sum_{i=1}^m |b_i(x) - \langle b_i \rangle_I| \mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m)(x) \\ & \quad + \left| \sum_{Q \in \mathcal{S}} \left( \sum_{i=1}^m |\langle b_i \rangle_I - \langle b_i \rangle_Q| \right) \prod_{j=1}^m \langle f_j \rangle_Q \chi_Q(x) - c \right|. \end{aligned}$$

Therefore, Let  $c_1 = \sum_{Q \in \mathcal{S}, Q \supset I} \left( \sum_{i=1}^m |\langle b_i \rangle_I - \langle b_i \rangle_Q| \right) \prod_{j=1}^m \langle f_j \rangle_Q$ , we thus have that

$$\begin{aligned} & \inf_{c \in \mathbb{C}} \left( \frac{1}{|I|} \int_I |\mathcal{A}_{m; \mathcal{S}, \bar{b}}(f_1, \dots, f_m)(x) - c|^{\delta} dx \right)^{\frac{1}{\delta}} \\ & \lesssim \left( \frac{1}{|I|} \int_I \left| \sum_{i=1}^m |b_i(x) - \langle b_i \rangle_I| \mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m)(x) \right|^{\delta} dx \right)^{\frac{1}{\delta}} \\ & \quad + \left( \frac{1}{|I|} \int_I \left| \sum_{Q \in \mathcal{S}} \left( \sum_{i=1}^m |\langle b_i \rangle_I - \langle b_i \rangle_Q| \right) \prod_{j=1}^m \langle f_j \rangle_Q \chi_Q(x) - c_1 \right|^{\delta} dx \right)^{\frac{1}{\delta}} \end{aligned}$$

Let  $\gamma \in (\delta, \frac{1}{m})$ . It follows from Hölder's inequality that

$$\begin{aligned} & \left( \frac{1}{|I|} \int_I \left| \sum_{i=1}^m |b_i(x) - \langle b_i \rangle_I| \mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m)(x) \right|^{\delta} dx \right)^{\frac{1}{\delta}} \\ & \lesssim \left( \frac{1}{|I|} \int_I |\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m)(x)|^{\gamma} dx \right)^{\frac{1}{\gamma}} \\ & \lesssim \inf_{y \in I} M_{\gamma}(\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m))(y). \end{aligned}$$

On the other hand, we deduce from Hölder's inequality, Lemma 4.2 and Lemma 4.3, that

$$\begin{aligned}
& \left( \frac{1}{|I|} \int_I \left| \sum_{Q \in \mathcal{S}, Q \subset I} \left( \sum_{i=1}^m |\langle b_i \rangle_I - \langle b_i \rangle_Q| \right) \prod_{j=1}^m \langle f_j \chi_I \rangle_Q \chi_Q(x) \right|^\delta dx \right)^{\frac{1}{\delta}} \\
& \lesssim \left( \frac{1}{|I|} \int_I (\mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1 \chi_I, \dots, f_m \chi_I)(x))^\delta dx \right)^{\frac{1}{\delta}} \\
& \quad + \left( \frac{1}{|I|} \int_I \left( \sum_{i=1}^m |b_i(x) - \langle b_i \rangle_I| \right) (\mathcal{A}_{m; \mathcal{S}}(f_1 \chi_I, \dots, f_m \chi_I)(x))^\delta dx \right)^{\frac{1}{\delta}} \\
& \lesssim \prod_{j=1}^m \langle |f_j| \rangle_I.
\end{aligned}$$

Combining the estimates above leads to (4.2).  $\square$

Let  $\mathcal{D}$  be a dyadic grid. Associated with  $\mathcal{D}$ , define the sharp maximal function  $M_{\mathcal{D}}^\sharp$  as

$$M_{\mathcal{D}}^\sharp f(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For  $\delta \in (0, 1)$ , let  $M_{\mathcal{D}, \delta}^\sharp f(x) = [M_{\mathcal{D}}^\sharp(|f|^\delta)(x)]^{1/\delta}$ . Repeating the argument in [38, p. 153], we can verify that if  $u \in A_\infty(\mathbb{R}^n)$  and  $\Phi$  is a increasing function on  $[0, \infty)$  which satisfies that

$$\Phi(2t) \leq C\Phi(t), \quad t \in [0, \infty),$$

then

$$(4.4) \quad \sup_{\lambda > 0} \Phi(\lambda) u(\{x \in \mathbb{R}^n : |h(x)| > \lambda\}) \lesssim \sup_{\lambda > 0} \Phi(\lambda) u(\{x \in \mathbb{R}^n : M_{\mathcal{D}, \delta}^\sharp h(x) > \lambda\}),$$

provided that  $\sup_{\lambda > 0} \Phi(\lambda) u(\{x \in \mathbb{R}^n : M_{\mathcal{D}, \delta} h(x) > \lambda\}) < \infty$ .

*Proof of Theorem 1.9.* Let  $\vec{\beta}_1 = (\frac{1}{s_1}, 0, \dots, 0)$ ,  $\dots$ ,  $\vec{\beta}_m = (0, \dots, 0, \frac{1}{s_m})$ . By the inequality (3.7) and the one-third trick, it suffices to prove that for  $\vec{w} = (w_1, \dots, w_m) \in A_{1, \dots, 1}(\mathbb{R}^{mn})$ ,  $i = 1, \dots, m$ , dyadic grid  $\mathcal{D}$  and sparse family  $\mathcal{S} \subset \mathcal{D}$ ,

$$\begin{aligned}
(4.5) \quad & \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}_i}}(f_1, \dots, f_m)(x) > 1\}) \\
& \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} |f_j(y_j)| \log^{\frac{1}{s_i}} (1 + |f_j(y_j)|) w_j(y_j) dy_j \right)^{\frac{1}{m}},
\end{aligned}$$

and

$$(4.6) \quad \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m)(x) > 1\}) \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}^{\frac{1}{m}}.$$

We first prove (4.5). By a standard limit argument, it suffices to consider the case that the sparse family  $\mathcal{S}$  is finite. Let  $\delta \in (0, \frac{1}{m})$ . The estimate (4.2) in Lemma 4.4 tells us that

$$M_{\mathcal{D}, \delta}^\sharp (\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}_i}}(f_1, \dots, f_m))(x) \lesssim \mathcal{M}_{L(\log L)^{\vec{\beta}_i}}(f_1, \dots, f_m)(x).$$

Let  $\psi_i(t) = t^{\frac{1}{m}} \log^{-\frac{1}{s_i}}(1 + t^{-\frac{1}{m}})$ . Lemma 4.1 now tells us that

$$\begin{aligned} & \sup_{t>0} \psi_i(t) \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{M}_{L(\log L)^{\vec{\beta}_i}}(f_1, \dots, f_m)(x) > t\}) \\ & \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} |f_j(y_j)| \log^{\frac{1}{s_i}}(1 + |f_j(y_j)|) w_j(y_j) dy_j \right)^{\frac{1}{m}}. \end{aligned}$$

This, via (4.4) and Lemma 4.1, implies that

$$\begin{aligned} & \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}_i}}(f_1, \dots, f_m)(x) > 1\}) \\ & \lesssim \sup_{t>0} \psi_i(t) \nu_{\vec{w}}(\{x \in \mathbb{R}^n : M_{\mathcal{D}, \delta}^{\sharp}(\mathcal{A}_{m; \mathcal{S}, L(\log L)^{\vec{\beta}_i}}(f_1, \dots, f_m))(x) > t\}) \\ & \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} |f_j(y_j)| \log^{\frac{1}{s_i}}(1 + |f_j(y_j)|) w_j(y_j) dy_j \right)^{\frac{1}{m}}. \end{aligned}$$

We turn our attention to (4.6). Again we assume that the sparse family  $\mathcal{S}$  is finite. Applying Lemma 4.4, we see that for  $\delta, \gamma$  with  $0 < \delta < \gamma < \frac{1}{m}$ ,

$$M_{\mathcal{D}, \delta}^{\sharp}(\mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m))(x) \lesssim M_{\gamma}(\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m))(x) + \mathcal{M}(f_1, \dots, f_m)(x).$$

Recalling that  $\nu_{\vec{w}} \in A_{\infty}(\mathbb{R}^{mn})$ , we can choose  $\delta$  and  $\gamma$  in (4.3) small enough such that  $\nu_{\vec{w}} \in A_{\frac{1}{m\gamma}}(\mathbb{R}^{mn})$ . It then follows from Lemma 3.2, the inequality (4.2) and Lemma 4.1 that

$$\begin{aligned} & \lambda^{\frac{1}{m}} \nu_{\vec{w}}(\{x \in \mathbb{R}^n : M_{\gamma}(\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m))(x) > \lambda\}) \\ & \lesssim \sup_{t>0} t^{\frac{1}{m}} \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m)(x) > t\}) \\ & \lesssim \sup_{t>0} t^{\frac{1}{m}} \nu_{\vec{w}}(\{x \in \mathbb{R}^n : M_{\mathcal{D}, \delta}^{\sharp}(\mathcal{A}_{m; \mathcal{S}}(f_1, \dots, f_m))(x) > t\}) \\ & \lesssim \sup_{t>0} t^{\frac{1}{m}} \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{M}(f_1, \dots, f_m)(x) > t\}) \\ & \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}^{\frac{1}{m}}, \end{aligned}$$

This, together with (4.4), leads to that

$$\begin{aligned} & \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m)(x) > 1\}) \\ & \lesssim \sup_{t>0} t^{\frac{1}{m}} \nu_{\vec{w}}(\{x \in \mathbb{R}^n : M_{\mathcal{D}, \delta}^{\sharp}(\mathcal{A}_{m; \mathcal{S}, \vec{b}}(f_1, \dots, f_m))(x) > t\}) \\ & \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n, w_j)}^{\frac{1}{m}}, \end{aligned}$$

and then completes the proof of Theorem 1.9.  $\square$

## 5. APPLICATIONS TO THE COMMUTATORS OF CALDERÓN

Let us consider the  $m$ -th commutator of Calderón, which is defined by

$$\mathcal{C}_{m+1}(a_1, \dots, a_m, f)(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{\prod_{j=1}^m (A_j(x) - A_j(y))}{(x - y)^{m+1}} f(y) dy,$$

where  $a_j = A'_j$ . This operator first appeared in the study of Cauchy integrals along Lipschitz curves and, in fact, led to the first proof of the  $L^2$  boundedness of the latter.

When  $m = 1$ , it is well known that  $\mathcal{C}_2$  is bounded from  $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$  to  $L^p(\mathbb{R})$  when  $1 < p_1, p_2 \leq \infty$  and  $\frac{1}{2} < p \leq \infty$  satisfying  $1/p = 1/p_1 + 1/p_2$ ; and moreover, it is bounded from  $L^{p_1}(R) \times L^{p_2}(\mathbb{R})$  to  $L^{p,\infty}(\mathbb{R})$  if  $\min\{p_1, p_2\} = 1$  and in particular it is bounded from  $L^1(\mathbb{R}) \times L^1(\mathbb{R})$  to  $L^{\frac{1}{2}}(\mathbb{R})$ ; see [4, 5]. The corresponding result that  $\mathcal{C}_3$  maps  $L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^1(\mathbb{R})$  to  $L^{\frac{1}{3},\infty}(\mathbb{R})$  was proved by Coifman and Meyer; see [9], while the analogous result for  $\mathcal{C}_{m+1}$ ,  $m \geq 3$ , was established by Duong, Grafakos, and Yan [14]. As it was proved in [14],  $\mathcal{C}_{m+1}$  can be rewritten as the following multilinear singular integral operator

$$(5.1) \quad \begin{aligned} & \mathcal{C}_{m+1}(a_1, \dots, a_m, f)(x) \\ &= \int_{\mathbb{R}^{m+1}} K(x; y_1, \dots, y_{m+1}) \prod_{j=1}^m a_j(y_j) f(y_{m+1}) dy_1 \dots dy_{m+1}; \end{aligned}$$

with

$$K(x; y_1, \dots, y_{m+1}) = \frac{(-1)^{me(y_{m+1}-x)}}{(x - y_{m+1})^{m+1}} \prod_{j=1}^m \chi_{(\min\{x, y_{m+1}\}, \max\{x, y_{m+1}\})}(y_j),$$

and  $e$  is the characteristic function of  $[0, \infty)$ . Using some new maximal operators, Grafakos, Liu and Yang [17] proved that if  $p_1, \dots, p_{m+1} \in [1, \infty)$  and  $p \in [\frac{1}{m+1}, \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_{m+1}$ , and  $\vec{w} = (w_1, \dots, w_m, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$ , then  $\mathcal{C}_{m+1}$  is bounded from  $L^{p_1}(\mathbb{R}, w_1) \times \dots \times L^{p_{m+1}}(\mathbb{R}, w_{m+1})$  to  $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$ , and when  $\min_{1 \leq j \leq m+1} p_j > 1$ ,  $\mathcal{C}_{m+1}$  is bounded from  $L^{p_1}(\mathbb{R}, w_1) \times \dots \times L^{p_{m+1}}(\mathbb{R}, w_{m+1})$  to  $L^p(\mathbb{R}, \nu_{\vec{w}})$ . It was pointed out in [22] that  $\mathcal{C}_{m+1}$  satisfies Assumption 1.4 and (1.11). Thus by Theorems 1.5, 1.8 and 1.9, we have the following conclusions.

**Corollary 5.1.** *Let  $m \geq 1$ ,  $p_1, \dots, p_{m+1}, q_1, \dots, q_{m+1} \in (1, \infty)$ ,  $p, q \in (\frac{1}{m+1}, \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_{m+1}$ ,  $1/q = 1/q_1 + \dots + 1/q_{m+1}$ ,  $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$ . Then*

$$\begin{aligned} & \|\{\mathcal{C}_{m+1}(a_1^k, \dots, a_m^k, f^k)\}\|_{L^p(l^q; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_{m+1}}{p}\}} \\ & \times \prod_{j=1}^m \|\{a_j^k\}\|_{L^{p_j}(l^{q_j}; \mathbb{R}, w_j)} \|\{f^k\}\|_{L^{p_{m+1}}(l^{q_{m+1}}; \mathbb{R}, w_{m+1})}. \end{aligned}$$

**Corollary 5.2.** *Let  $m \geq 1$ ,  $p_1, \dots, p_{m+1}, q_1, \dots, q_{m+1} \in (1, \infty)$ ,  $p, q \in (\frac{1}{m+1}, \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_{m+1}$ ,  $1/q = 1/q_1 + \dots + 1/q_{m+1}$ ,  $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$ . Let  $b_j \in \text{Osc}_{\exp L^{s_j}}(\mathbb{R})$  with  $\sum_{j=1}^{m+1} \|b_j\|_{\text{Osc}_{\exp L^{s_j}}(\mathbb{R})} = 1$ . Then  $\mathcal{C}_{m+1, \vec{b}}$ , the commutator of  $\mathcal{C}_{m+1}$  defined as (1.12), satisfies the weighted estimate that*

$$\begin{aligned} & \|\{\mathcal{C}_{m+1, \vec{b}}(a_1^k, \dots, a_m^k, f^k)\}\|_{L^p(l^q; \mathbb{R}^n, \nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p'_1}{p}, \dots, \frac{p'_{m+1}}{p}\}} \\ & \times \left( [\nu_{\vec{w}}]_{A_{\infty}}^{\frac{1}{s_*}} + \sum_{i=1}^m [\sigma_i]_{A_{\infty}}^{\frac{1}{s_i}} \right) \prod_{j=1}^m \|\{a_j^k\}\|_{L^{p_j}(l^{q_j}; \mathbb{R}, w_j)} \|\{f^k\}\|_{L^{p_{m+1}}(l^{q_{m+1}}; \mathbb{R}, w_{m+1})}. \end{aligned}$$

**Corollary 5.3.** *Let  $m \geq 1$ ,  $q_1, \dots, q_{m+1} \in (1, \infty)$ ,  $q \in (1/(m+1), \infty)$  with  $1/p = 1/p_1 + \dots + 1/p_{m+1}$ ,  $1/q = 1/q_1 + \dots + 1/q_{m+1}$ ,  $\vec{w} = (w_1, \dots, w_{m+1}) \in A_{\vec{P}}(\mathbb{R}^{m+1})$ .*

Let  $b_j \in \text{Osc}_{\exp L^{s_j}}(\mathbb{R})$  ( $1 \leq j \leq m+1$ ) with  $\sum_{j=1}^{m+1} \|b_j\|_{\text{Osc}_{\exp L^{s_j}}(\mathbb{R})} = 1$ . Then for each  $\lambda > 0$ ,

$$\begin{aligned} & \nu_{\vec{w}}(\{x \in \mathbb{R}^n : \|\{\mathcal{C}_{m+1, \vec{b}}(a_1^k, \dots, a_m^k, f^k)(x)\}_{l^q} > \lambda\}) \\ & \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \frac{\|\{a_j^k(y_j)\}_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}} \log^{\frac{1}{s_*}} \left( 1 + \frac{\|\{a_j^k(y_j)\}_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}} \right) w_j(y_j) dy_j \right)^{\frac{1}{m+1}} \\ & \quad \times \left( \int_{\mathbb{R}^n} \frac{\|\{f^k(y)\}_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}} \log^{\frac{1}{s_*}} \left( 1 + \frac{\|\{f^k(y)\}_{l^{q_j}}}{\lambda^{\frac{1}{m+1}}} \right) w_{m+1}(y) dy \right)^{\frac{1}{m+1}}. \end{aligned}$$

**Added in Proof.** After this paper was prepared, we learned that Dr. Kangwei Li [29] also observed that, Lerner's idea in [26] applies to the multilinear singular integral operators. We remark that our argument in the proof of Theorem 3.4 also based on this observation. Li [29] proved that the multilinear singular integral operators whose kernels satisfy  $L^r$ -Hörmander condition can be dominated by sparse operators. The main results in [29] are different from the results in this paper and are of independent interest.

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